

ON REPRESENTATION OF AN INTEGER BY $X^2 + Y^2 + Z^2$ AND THE MODULAR EQUATIONS OF DEGREE 3 AND 5.

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There are always flowers for those who want to see them

ABSTRACT. I discuss a variety of results involving $s(n)$, the number of representations of n as a sum of three squares. One of my objectives is to reveal numerous interesting connections between the properties of this function and certain modular equations of degree 3 and 5. In particular, I show that

$$s(25n) = (6 - (-n|5))s(n) - 5s\left(\frac{n}{25}\right)$$

follows easily from the well known Ramanujan modular equation of degree 5. Moreover, I establish new relations between $s(n)$ and $h(n)$, $g(n)$, the number of representations of n by the ternary quadratic forms

$$2x^2 + 2y^2 + 2z^2 - yz + zx + xy, \quad x^2 + y^2 + 3z^2 + xy,$$

respectively. I also find generating function formulae for various subsequences of $\{s(n)\}$, for instance

$$6 \prod_{j=1}^{\infty} (1 - q^{2j})^2 (1 - q^{10j}) (1 + q^{-1+2j})^4 (1 + q^{-3+10j}) (1 + q^{-7+10j}) = \sum_{n=0}^{\infty} s(5n+1)q^n.$$

I propose an interesting identity for $s(p^2n) - ps(n)$ with p being an odd prime.

1. INTRODUCTION

Let $(a, b, c, d, e, f)(n)$ denote the number of representations of n by ternary form $ax^2 + by^2 + cz^2 + dyz + czx + fxy$. I will assume that $(a, b, c, d, e, f)(n) = 0$, whenever $n \notin Z$. Let $s(n)$ denote the number of representations of n by ternary form $x^2 + y^2 + z^2$. In a path-blazing paper [12], Hirschhorn and Sellers proved in a completely elementary manner that

$$(1.1) \quad s(p^2n) = (p + 1 - (-n|p))s(n) - ps\left(\frac{n}{p^2}\right),$$

when $p = 3$. Here $(a|p)$ denotes the Legendre symbol. It should be pointed out that the authors of [12] proved (1.1) for all odd prime numbers p by an appeal to

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the theory of modular forms. In Section 2, I will show that (1.1) with $p = 5$ follows easily from the well-known identity for $\phi(q)^2 - \phi(q^5)^2$ with

$$(1.2) \quad \phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

Here and throughout, q is a complex number with $|q| < 1$. I will also provide an elementary proof of the following

Theorem 1.1. *If $n \equiv 1, 2 \pmod{4}$, then*

$$(1.3) \quad s(25n) - 5s(n) = 4(2, 2, 2, -1, 1, 1)(n),$$

and

Theorem 1.2. *If $n \equiv 1, 2 \pmod{4}$, then*

$$(1.4) \quad s(9n) - 3s(n) = 2(1, 1, 3, 0, 0, 1)(n).$$

In Section 5, I will show how to remove the parity restrictions in the above theorems. But then there is a price to pay. It is not an exorbitant price, however. Let me point out that two ternary forms $2x^2 + 2y^2 + 2z^2 - yz + zx + xy$ and $x^2 + y^2 + 3z^2 + xy$ both have class number one. This implies that these forms are both regular [9], [15], [14]. For a recent discussion of the relation between the Ramanujan modular equations and certain ternary quadratic forms the reader is invited to examine [2]. And it goes without saying that one should not forget the timeless classic [1].

I begin by recalling some standard notations and definitions

$$(1.5) \quad (a; q)_{\infty} := \prod_{j \geq 0} (1 - aq^j),$$

and

$$(1.6) \quad E(q) := \prod_{j \geq 1} (1 - q^j).$$

Ramanujan's general theta-function $f(a, b)$ is defined by

$$(1.7) \quad f(a, b) = \sum_{n=-\infty}^{\infty} a^{\frac{(n-1)n}{2}} b^{\frac{(n+1)n}{2}}, \quad |ab| < 1.$$

In Ramanujan's notation, the celebrated Jacobi triple product identity takes the shape [4], p.35

$$(1.8) \quad f(a, b) = (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}, \quad |ab| < 1.$$

Note that $\phi(q)$ can be interpreted as

$$(1.9) \quad \phi(q) = f(q, q) = \frac{E(q^2)^5}{E(q^4)^2 E(q)^2},$$

where the product on the right follows easily from (1.7). We shall also define

$$(1.10) \quad \psi(q) = f(q, q^3).$$

It is not hard to check that

$$(1.11) \quad \psi(q) = \frac{1}{2} f(1, q) = \sum_{n \geq 0} q^{\frac{(n+1)n}{2}} = \frac{E(q^2)^2}{E(q)},$$

and that

$$(1.12) \quad f(q, q^9)f(q^3, q^7) = \frac{E(q^{20})E(q^5)E(q^2)^2}{E(q^4)E(q)},$$

$$(1.13) \quad f(q, q^4)f(q^2, q^3) = \frac{E(q^5)^2E(q^2)}{E(q^{10})E(q)}.$$

Function $f(a, b)$ may be dissected in many different ways. We will use the following dissections [4], p.40, p.49

$$(1.14) \quad \phi(q) = \phi(q^4) + 2q\psi(q^8),$$

$$(1.15) \quad \phi(q) = \phi(q^9) + 2qf(q^3, q^{15}),$$

$$(1.16) \quad \phi(q) = \phi(q^{25}) + 2qf(q^{15}, q^{35}) + 2q^4f(q^5, q^{45}).$$

We will also require a special case of Schroter's formula [4], p. 45

$$(1.17) \quad f(a, b)f(c, d) = f(ac, bd)f(ad, bc) + af\left(\frac{b}{c}, ac^2d\right)f\left(\frac{b}{d}, acd^2\right),$$

provided $ab = cd$. Setting $a = b = c = d = q$ in (1.17) we obtain

$$(1.18) \quad \phi(q)^2 = \phi(q^2)^2 + 4q\psi(q^4)^2.$$

Iterating, we find that

$$(1.19) \quad \phi(q)^2 = \phi(q^4)^2 + 4q\psi(q^4)^2 + 4q^2\psi(q^8)^2.$$

Next, we set $a = q, b = q^9, c = q^3, d = q^7$ in (1.17) and square the result. This way we have

$$(1.20) \quad \begin{aligned} f(q, q^9)^2 f(q^3, q^7)^2 &= f(q^4, q^{16})^2 f(q^8, q^{12})^2 \\ &+ 2qf(q^4, q^{16})f(q^8, q^{12})f(q^6, q^{14})f(q^2, q^{18}) + q^2f(q^6, q^{14})^2 f(q^2, q^{18})^2. \end{aligned}$$

Finally, we multiply both sides in (1.20) by

$$\frac{E(q^4)\phi(q^5)}{E(q^{20})E(q^{10})^2},$$

and use (1.9), (1.11), (1.12) and (1.13) to arrive at

$$(1.21) \quad \begin{aligned} \phi(q)f(q^2, q^8)f(q^4, q^6) &= \psi(q^4)\phi(q^5)\phi(q^{10}) \\ &+ 2q\psi(q^2)\psi(q^{10})\phi(q^5) + q^2\psi(q^{20})\phi(q^2)\phi(q^5). \end{aligned}$$

This result will come in handy in my proof of (1.3) with $n \equiv 2 \pmod{4}$. To deal with the case $n \equiv 1 \pmod{4}$ in (1.3) I will need another identity

$$(1.22) \quad \phi(q)\phi(q^5) + \sum_{m,n} q^{2m^2+2nm+3n^2} = 2 \frac{E(q^{10})E(q^5)E(q^4)E(q^2)}{E(q^{20})E(q)}.$$

This formula was discovered and proven in [3]. The proof of (1.22), given in [3], used only a special case of the Ramanujan ${}_1\psi_1$ summation formula [5], p.64. Multiplying both sides in (1.22) by $\psi(q^{10})$ and utilizing (1.11) and (1.12) we can rewrite (1.22) as

$$(1.23) \quad \psi(q^{10})\phi(q)\phi(q^5) + \psi(q^{10}) \sum_{m,n} q^{2m^2+2nm+3n^2} = 2\psi(q^2)f(q, q^9)f(q^3, q^7).$$

2. THE TERNARY IMPLICATIONS OF THE FUNDAMENTAL MODULAR EQUATION OF DEGREE 5

In this section we will make an extensive use of a well-known modular equation of degree 5

$$(2.1) \quad \phi(q)^2 - \phi(q^5)^2 = 4qf(q, q^9)f(q^3, q^7)$$

to prove (1.1) with $p = 5$. We note that (2.1) has an attractive companion

$$(2.2) \quad \psi(q)^2 - q\psi(q^5)^2 = f(q, q^4)f(q^2, q^3).$$

Both (2.1), (2.2) are discussed in [4], p.262. We remark that the right hand side of (2.1) was interpreted in terms of so-called self-conjugate 5-cores in [10]. To proceed further I will need a sifting operator $S_{t,s}$. It is defined by its action on power series as follows

$$(2.3) \quad S_{t,s} \sum_{n \geq 0} c(n)q^n = \sum_{k \geq 0} c(tk + s)q^k.$$

Here t, s are integers such that $0 \leq s < t$. Making use of (1.16), we find that

$$(2.4) \quad S_{5,0}\phi(q)^2 = \phi(q^5)^2 + 8qf(q, q^9)f(q^3, q^7).$$

And so

$$(2.5) \quad S_{5,0}(\phi(q)^2 - \phi(q^5)^2) = -(\phi(q)^2 - \phi(q^5)^2) + 8qf(q, q^9)f(q^3, q^7).$$

Employing (2.1) twice, we see that

$$(2.6) \quad S_{5,0}(qf(q, q^9)f(q^3, q^7)) = qf(q, q^9)f(q^3, q^7).$$

Analogously, we can check that

$$(2.7) \quad S_{5,0}\phi(q)^3 = \phi(q^5)^3 + 24q\phi(q^5)f(q, q^9)f(q^3, q^7),$$

and that

$$(2.8) \quad S_{5,1}\phi(q)^3 = 6f(q^3, q^7)(\phi(q^5)^2 + 4qf(q, q^9)f(q^3, q^7)) = 6f(q^3, q^7)\phi(q)^2,$$

$$(2.9) \quad S_{5,4}\phi(q)^3 = 6f(q, q^9)(\phi(q^5)^2 + 4qf(q, q^9)f(q^3, q^7)) = 6f(q, q^9)\phi(q)^2.$$

Thanks to (1.8), the right hand side in (2.8) can be rewritten as an infinite product. This way we get

$$\sum_{n=0}^{\infty} s(5n+1)q^n = 6 \prod_{j=1}^{\infty} (1 - q^{2j})^2 (1 - q^{10j}) (1 + q^{-1+2j})^4 (1 + q^{-3+10j}) (1 + q^{-7+10j}).$$

With the aid of (1.16) we can combine (2.8) and (2.9) into a single elegant statement

$$(2.10) \quad S_{5,r}(\phi(q)^3 - 3\phi(q)\phi(q^5)^2) = 0,$$

where $r = 1, 4$. Next, we hit both sides of (2.7) with $S_{5,0}$ to obtain, with a little help from (2.6)

$$(2.11) \quad S_{25,0}\phi(q)^3 = \phi(q)^3 + 24q\phi(q)f(q, q^9)f(q^3, q^7).$$

Subtracting $5\phi(q)^3$ and making use of (2.1) again, we deduce that

$$(2.12) \quad \begin{aligned} S_{25,0}\phi(q)^3 - 5\phi(q)^3 &= -4\phi(q)^3 + 6\phi(q)(\phi(q)^2 - \phi(q^5)^2) \\ &= 2(\phi(q)^3 - 3\phi(q)\phi(q^5)^2). \end{aligned}$$

Finally, we apply $S_{5,r}$ with $r = 1, 4$ to both sides of (2.12) to find that

$$(2.13) \quad S_{125,25r}\phi(q)^3 - 5S_{5,r}\phi(q)^3 = 0.$$

But it is plain that

$$(2.14) \quad \phi(q)^3 = \sum_{n=0}^{\infty} s(n)q^n.$$

And so the equation (2.13) can be interpreted as

$$(2.15) \quad s(25n) - 5s(n) = 0,$$

when $n \equiv 1, 4 \pmod{5}$. Thus, the proof of (1.1) with $p = 5$ and $n \equiv 1, 4 \pmod{5}$ is complete. We now turn our attention to the $n \equiv 2, 3 \pmod{5}$ case. Subtracting $2\phi(q)^3$ from the extremes of (2.12), we end up with the formula

$$(2.16) \quad S_{25,0}\phi(q)^3 - 7\phi(q)^3 = -6\phi(q)\phi(q^5)^2.$$

It is now clear that for $r = 2, 3$

$$(2.17) \quad S_{5,r}(S_{25,0}\phi(q)^3 - 7\phi(q)^3) = -6\phi(q)^2 S_{5,r}\phi(q) = 0,$$

where in the last step we took advantage of the dissection formula (1.16). Obviously, (2.17) is equivalent to

$$(2.18) \quad s(25n) - 7s(n) = 0,$$

when $n \equiv 2, 3 \pmod{5}$. And so we completed the proof of (1.1) with $p = 5$ and $n \equiv 2, 3 \pmod{5}$. All that remains to do is to take care of the $n \equiv 0 \pmod{5}$ case. Adding $\phi(q)^3$ to both sides of (2.16) and applying $S_{5,0}$ to the result, we get

$$(2.19) \quad S_{5,0}(S_{25,0}\phi(q)^3 - 6\phi(q)^3) = S_{5,0}(\phi(q)^3 - 6\phi(q)\phi(q^5)^2).$$

Next, we utilize (1.16), (2.1) and (2.7) to process the right hand side of (2.19) as follows

$$\begin{aligned} S_{5,0}(\phi(q)^3 - 6\phi(q)\phi(q^5)^2) &= \phi(q^5)^3 + 6\phi(q^5)(\phi(q)^2 - \phi(q^5)^2) - 6\phi(q^5)\phi(q)^2 \\ &= -5\phi(q^5)^3. \end{aligned}$$

Hence, we have shown that

$$(2.20) \quad S_{125,0}\phi(q)^3 - 6S_{5,0}\phi(q)^3 = -5\phi(q^5)^3.$$

Consequently,

$$(2.21) \quad s(25n) - 6s(n) = -5s\left(\frac{n}{25}\right),$$

when $5|n$. This concludes our proof of (1.1) with $p = 5$.

3. PROOF OF THE THEOREM 1.1

I begin by observing that the Theorem 1.1 is equivalent to the following statement

$$(3.1) \quad S_{100,25r}\phi(q)^3 - 5S_{4,r}\phi(q)^3 = 4S_{4,r}T(q),$$

where

$$T(q) := \sum_{x,y,z} q^{2x^2+2y^2+2z^2-yz+zx+xy}$$

and $r = 1, 2$. It is not hard to verify that

$$(3.2) \quad S_{4,1}T(q) = 6S_{4,1}X(1, q),$$

and that

$$(3.3) \quad S_{4,2}T(q) = 3S_{4,2}(X(0, q) + X(2, q)).$$

Here

$$(3.4) \quad X(r, q) := \sum_{\substack{x, \\ y \equiv -z \equiv r \pmod{4}}} q^{2x^2+2y^2+2z^2-yz+zx+xy}.$$

It takes little effort to check that

$$2x^2 + 2y^2 + 2z^2 - zy + zx + xy = 2 \left(x + \frac{y+z}{4} \right)^2 + \frac{10(3y^2 + 3z^2 - 2yz)}{16}.$$

Hence,

$$(3.5) \quad X(r, q) = \phi(q^2) \sum_{y,z} q^{30y^2+30z^2+20yz+20r(y+z)+5r^2}.$$

In addition,

$$10(y-z)^2 + 20(y+z+a)^2 = 30y^2 + 30z^2 + 20yz + 40a(y+z) + 20a^2.$$

And so

$$X(0, q) + X(2, q) = \phi(q^2) \left(\sum_{u \equiv v \pmod{2}} q^{10u^2+20v^2} + \sum_{u \not\equiv v \pmod{2}} q^{10u^2+20v^2} \right).$$

It is now evident that

$$(3.6) \quad X(0, q) + X(2, q) = \phi(q^2)\phi(q^{10})\phi(q^{20}).$$

Using this last result in (3.3), we find that

$$(3.7) \quad S_{4,2}T(q) = 3\phi(q^5)S_{4,2}(\phi(q^2)\phi(q^{10})).$$

Recalling (1.14), we obtain at once that

$$(3.8) \quad 4S_{4,2}T(q) = 24\phi(q^5)(\psi(q^4)\phi(q^{10}) + 6q^2\phi(q^2)\psi(q^{20})).$$

We now turn to (3.2). Thanks to (3.5) we can rewrite it as

$$(3.9) \quad 4S_{4,1}T(q) = 24S_{4,1} \left(\phi(q^2) \sum_{y,z} q^{30y^2+30z^2+20yz+20(y+z)+5} \right).$$

Next, we use a dissection (1.14) together with the obvious

$$30(u-v)^2 + 30(u+v)^2 + 20(u-v)(u+v) + 20(u+v+u-v) = 4(20u^2 + 10u + 10v^2),$$

$$\begin{aligned} 30(u-v)^2 + 30(u+v-1)^2 + 20(u-v)(u+v-1) + 20(u+v-1+u-v) + 2 \\ = 4(20u^2 - 10u + 10v^2 - 10v + 3) \end{aligned}$$

to process the expression on the right of (3.9) as

$$\begin{aligned}
 & 24q\phi(q^2)S_{4,0} \left(\sum_{y \equiv z \pmod{2}} q^{30y^2+30z^2+20yz+20(y+z)} \right) + \\
 & 48q\psi(q^4)S_{4,0} \left(\sum_{y \not\equiv z \pmod{2}} q^{30y^2+30z^2+20yz+20(y+z)+2} \right) = \\
 (3.10) \quad & 24q\phi(q^2) \sum_{u,v} q^{20u^2+10u+10v^2} + 48q^4\psi(q^4) \sum_{u,v} q^{20u^2-10u+10v^2-10v}.
 \end{aligned}$$

It is now transparent that

$$4S_{4,1}T(q) = 24q\psi(q^{10})(\phi(q^2)\phi(q^{10}) + 4q^3\psi(q^4)\psi(q^{20})).$$

Also, it is not hard to check that

$$\begin{aligned}
 \sum_{m,n} q^{2m^2+2nm+3n^2} &= \sum_{m,n} q^{2(m+n)^2+10n^2} + q^3 \sum_{m,n} q^{2(m+n+1)(m+n)+10(n+1)n} \\
 &= \phi(q^2)\phi(q^{10}) + 4q^3\psi(q^4)\psi(q^{20}).
 \end{aligned}$$

This implies that

$$(3.11) \quad 4S_{4,1}T(q) = 24q\psi(q^{10}) \sum_{m,n} q^{2m^2+2nm+3n^2}.$$

Next, we employ (2.12) to get

$$(3.12) \quad S_{100,25r}\phi(q)^3 - 5S_{4,r}\phi(q)^3 = 2S_{4,r}(\phi(q)^3 - 3\phi(q)\phi(q^5)^2).$$

With the aid of (1.14), (1.19), (2.1), (2.2) we verify that

$$(3.13) \quad S_{4,1}(\phi(q)^3 - 3\phi(q)\phi(q^5)^2) = 24q\psi(q^2)f(q, q^9)f(q^3, q^7) - 12q\phi(q)\phi(q^5)\psi(q^{10}),$$

$$(3.14) \quad S_{4,2}(\phi(q)^3 - 3\phi(q)\phi(q^5)^2) = -24q\psi(q^2)\psi(q^5)^2 + 12\phi(q)f(q^2, q^8)f(q^4, q^6).$$

Utilizing these results in (3.12) we obtain

$$(3.15) \quad S_{100,25}\phi(q)^3 - 5S_{4,1}\phi(q)^3 = 48q\psi(q^2)f(q, q^9)f(q^3, q^7) - 24q\phi(q)\phi(q^5)\psi(q^{10}),$$

$$(3.16) \quad S_{100,50}\phi(q)^3 - 5S_{4,2}\phi(q)^3 = -48q\psi(q^2)\psi(q^5)^2 + 24\phi(q)f(q^2, q^8)f(q^4, q^6).$$

Recalling (3.11), we see that (3.1) with $r = 1$ is equivalent to

$$2\psi(q^2)f(q, q^9)f(q^3, q^7) - \phi(q)\phi(q^5)\psi(q^{10}) = \psi(q^{10}) \sum_{m,n} q^{2m^2+2nm+3n^2},$$

which is, essentially, (1.23). Analogously, employing (3.8), we find that (3.1) with $r = 2$ is equivalent to

$$-2q\psi(q^2)\psi(q^5)^2 + \phi(q)f(q^2, q^8)f(q^4, q^6) = \phi(q^5)\psi(q^4)\phi(q^{10}) + 6q^2\phi(q^2)\phi(q^5)\psi(q^{20}),$$

which is, essentially, (1.21). The proof of the Theorem 1.1 is now complete.

4. CUBIC MODULAR IDENTITIES REVISITED.

As in the last section, I begin by observing that the Theorem 1.2 is equivalent to the following statement

$$(4.1) \quad S_{36,9r}\phi(q)^3 - 3S_{4,r}\phi(q)^3 = 4S_{4,r}\phi(q^3)a(q),$$

where

$$a(q) := \sum_{x,y} q^{x^2+xy+y^2},$$

and $r = 1, 2$. Function $a(q)$ was extensively studied in the literature [6], [7], [8], [11]. It appeared in Borwein's cubic analogue of Jacobi's celebrated theta function identity [7]. I will record below some useful formulas

$$(4.2) \quad 4a(q^2)\phi(q^3) = \phi(q)^3 + 3\frac{\phi(q^3)^4}{\phi(q)},$$

$$(4.3) \quad a(q) = a(q^3) + 6q\frac{E(q^9)^3}{E(q^3)},$$

$$(4.4) \quad a(q) = \phi(q)\phi(q^3) + 4q\psi(q^2)\psi(q^6),$$

$$(4.5) \quad a(q) = 2\phi(q)\phi(q^3) - \phi(-q)\phi(-q^3),$$

$$(4.6) \quad 2a(q^2) - a(q) = \frac{\phi(-q)^3}{\phi(-q^3)}$$

$$(4.7) \quad a(q) = a(q^4) + 6q\psi(q^2)\psi(q^6).$$

Formula (4.2) appears as equation (6.4) in [6]. Identities (4.3), (4.4), (4.5) and (4.6) are discussed in [8]. In order to prove (4.7), the authors of [11] have shown that

$$(4.8) \quad 2q\psi(q^2)\psi(q^6) = \sum_{u \not\equiv v \pmod{2}} q^{u^2+3v^2}.$$

We have at once that

$$(4.9) \quad \begin{aligned} 2q\psi(q^2)\psi(q^6) &= \sum_{\substack{u \equiv 1 \pmod{2}, \\ v \equiv 0 \pmod{2}}} q^{u^2+3v^2} + \sum_{\substack{u \equiv 0 \pmod{2}, \\ v \equiv 1 \pmod{2}}} q^{u^2+3v^2} \\ &= 2q\psi(q^8)\phi(q^{12}) + 2q^3\phi(q^4)\psi(q^{24}). \end{aligned}$$

Combining (4.7) and (4.9), we have a pretty neat dissection of $a(q) \pmod{4}$

$$(4.10) \quad a(q) = a(q^4) + 6q\psi(q^8)\phi(q^{12}) + 6q^3\phi(q^4)\psi(q^{24}).$$

In [16], L.C. Shen discussed two well-known modular identities of degree 3

$$(4.11) \quad \phi(q)^2 - \phi(q^3)^2 = 4q\frac{\psi(q)\psi(q^3)\psi(q^6)}{\psi(q^2)},$$

and

$$(4.12) \quad \phi(q)^2 + \phi(q^3)^2 = 2\frac{\psi(q)f(q, q^2)f(q^2, q^4)}{\psi(q^2)}.$$

Multiplying (4.11) and (4.12), and using

$$(4.13) \quad f(q, q^2) = \frac{E(q^3)^2 E(q^2)}{E(q^6) E(q)},$$

$$(4.14) \quad f(q, q^5) = \frac{E(q^{12})E(q^3)E(q^2)^2}{E(q^6)E(q^4)E(q)}$$

together with (1.11) we have

$$(4.15) \quad \phi(q)^4 - \phi(q^3)^4 = 8q\phi(q^3)f(q, q^5)^3.$$

Next, we rewrite (4.15) as

$$(4.16) \quad \frac{\phi(q)^4}{\phi(q^3)} = \phi(q^3)^3 + 8qf(q, q^5)^3.$$

Recalling (1.15), we can recognize the expression on the right as

$$\phi(q^3)^3 + 8qf(q, q^5)^3 = S_{3,0}(\phi(q^9) + 2qf(q^3, q^{15}))^3 = S_{3,0}\phi(q)^3.$$

And so

$$(4.17) \quad S_{3,0}\phi(q)^3 = \frac{\phi(q)^4}{\phi(q^3)}.$$

Thus, we derived a simple product expansion for the following generating function

$$\sum_{n=0}^{\infty} s(3n)q^n = \frac{E(q^{12})^2 E(q^3)^2 E(q^2)^{20}}{E(q^6)^5 E(q^4)^8 E(q)^8}.$$

Surprisingly, this result escaped attention of the authors of citeHS, who used (1.15) to deduce

$$(4.18) \quad S_{3,1}\phi(q)^3 = 6\phi(q^3)^2 f(q, q^5)$$

and

$$(4.19) \quad S_{3,2}\phi(q)^3 = 12\phi(q^3)f(q, q^5)^2 = 12\phi(q)\frac{E(q^6)^3}{E(q^2)}.$$

Comparing (4.3) and (4.19), we find that

$$(4.20) \quad S_{3,2}(\phi(q)^3 - 2\phi(q^3)a(q^2)) = 0.$$

This result be required shortly. We are now well equipped to establish the following curious

Corollary 4.1.

$$(4.21) \quad S_{3,0}\left(\frac{\phi(q^3)^4}{\phi(q)}\right) = \phi(q^3)^3,$$

$$(4.22) \quad S_{3,1}\left(\frac{\phi(q^3)^4}{\phi(q)}\right) = -2\phi(q^3)^2 f(q, q^5),$$

$$(4.23) \quad S_{3,2}\left(\frac{\phi(q^3)^4}{\phi(q)}\right) = 4\phi(q^3)f(q, q^5)^2.$$

For the sake of brevity I prove only (4.21) and leave (4.22) and (4.23) as an exercise for a motivated reader. Clearly, (4.2) implies that

$$(4.24) \quad 3S_{3,0}\left(\frac{\phi(q^3)^4}{\phi(q)}\right) = 4\phi(q)S_{3,0}a(q^2) - S_{3,0}\phi(q)^3.$$

Using (4.3) with q replaced by q^2 and (4.17) in (4.24) we find that

$$3S_{3,0}\left(\frac{\phi(q^3)^4}{\phi(q)}\right) = \phi(q)\frac{4a(q^2)\phi(q^3) - \phi(q)^3}{\phi(q^3)}.$$

Finally, we employ (4.2) again to complete the proof of (4.21). Next, we want to show that

$$(4.25) \quad S_{9,0}\phi(q)^3 = \frac{4\phi(q)^4 - 3\phi(q^3)^4}{\phi(q)}.$$

To this end, we apply $S_{3,0}$ to both sides of (4.17). Utilizing (1.15), we find that

$$(4.26) \quad S_{9,0}\phi(q)^3 = \frac{\phi(q^3)^4 + 4(8q\phi(q^3)f(q, q^5)^3)}{\phi(q)}.$$

The statement in (4.25) follows immediately from (4.15) and (4.26). Moreover, we have

$$(4.27) \quad S_{9,0}\phi(q)^3 - \phi(q)^3 = 24q \frac{\phi(q^3)f(q, q^5)^3}{\phi(q)} = 24q f(q, q^5) \frac{E(q^6)^3}{E(q^2)},$$

$$(4.28) \quad S_{9,0}\phi(q)^3 - 4\phi(q)^3 = -3 \frac{\phi(q^3)^4}{\phi(q)},$$

$$(4.29) \quad S_{9,0}\phi(q)^3 - 5\phi(q)^3 = -\phi(q)^3 - 3 \frac{\phi(q^3)^4}{\phi(q)} = -4a(q^2)\phi(q^3).$$

Combining (4.3) with q replaced by q^2 and (4.29), we find that

$$S_{27,9}\phi(q)^3 - 5S_{3,1}\phi(q)^3 = -4\phi(q)S_{3,1}a(q^2) = 0,$$

which implies that

$$s(9n) = 5s(n),$$

when $n \equiv 1 \pmod{3}$. Similarly, from (4.21) and (4.27) we obtain

$$S_{27,0}\phi(q)^3 - 4S_{3,0}\phi(q)^3 = -3S_{3,0} \frac{\phi(q^3)^4}{\phi(q)} = -3\phi(q^3)^3.$$

Hence,

$$s(9n) = 4s(n) - 3s\left(\frac{n}{3}\right),$$

when $n \equiv 0 \pmod{3}$. Adding $2\phi(q)^3$ to the extremes in (4.29) we derive

$$(4.30) \quad S_{9,0}\phi(q)^3 - 3\phi(q)^3 = 2\phi(q)^3 - 4a(q^2)\phi(q^3).$$

Applying $S_{3,2}$ to both sides of (4.30), we obtain with the aid of (4.20)

$$S_{27,18}\phi(q)^3 - 3S_{3,2}\phi(q)^3 = 2S_{3,2}(\phi(q)^3 - 2\phi(q^3)a(q^2)) = 0.$$

Thus, we have shown that

$$s(9n) = 3s(n),$$

when $n \equiv 2 \pmod{3}$.

5. PROOF OF THE THEOREM 1.2 Y MUCHO MAS.

I begin this section by providing an easy proof of two formulas in (4.1). All I need is the following

Lemma 5.1. *If $r = 1, 2$, then*

$$(5.1) \quad S_{4,r}(\phi(q)^3 - 2a(q^2)\phi(q^3)) = S_{4,r}(a(q)\phi(q^3)).$$

Proof: This lemma is a straightforward corollary of (1.14), (4.7) and (4.10). Next, we apply $S_{4,r}$ with $r = 1, 2$ to (4.30) and use (5.1) to obtain

$$(5.2) \quad S_{36,9r}\phi(q)^3 - 3S_{4,r}\phi(q)^3 = 2S_{4,r}(\phi(q)^3 - 2\phi(q^3)a(q^2)) = 2S_{4,r}(a(q)\phi(q^3)),$$

which is (4.1), as desired. The proof of the Theorem 1.2 is now complete. We can do much better, if we realize that (5.1) is an immediate consequence of the following elegant result

$$(5.3) \quad \phi(q)^3 = \phi(q^3)(a(q) + 2a(q^2) - 2a(q^4)).$$

To prove it, we divide both sides by $\phi(q^3)$ and obtain

$$(5.4) \quad \frac{\phi(q)^3}{\phi(q^3)} = 2a(q^2) - a(q) + 2(a(q) - a(q^4)).$$

Using (4.6) and (4.7) in (5.4), we see that (5.3) is equivalent to

$$(5.5) \quad \frac{\phi(q)^3}{\phi(q^3)} - \frac{\phi(-q)^3}{\phi(-q^3)} = 12q\psi(q^2)\psi(q^6).$$

To verify (5.5), I replace q by $-q$ in (4.6) and subtract (4.6) to find with the aid of (4.5) the following

$$(5.6) \quad \frac{\phi(q)^3}{\phi(q^3)} - \frac{\phi(-q)^3}{\phi(-q^3)} = a(q) - a(-q) = 3(\phi(q)\phi(q^3) - \phi(-q)\phi(-q^3)).$$

Subtracting (4.4) from (4.5) we obtain

$$(5.7) \quad \phi(q)\phi(q^3) - \phi(-q)\phi(-q^3) = 4q\psi(q^2)\psi(q^6).$$

Hence,

$$(5.8) \quad \frac{\phi(q)^3}{\phi(q^3)} - \frac{\phi(-q)^3}{\phi(-q^3)} = 12q\psi(q^2)\psi(q^6),$$

as desired. This completes the proof of (5.3). We are now in a position to improve on (5.2). Indeed, it follows from (4.30) and (5.3) that

$$(5.9) \quad S_{9,0}\phi(q)^3 - 3\phi(q)^3 = 2\phi(q^3)a(q) - 4\phi(q^3)a(q^4).$$

Consequently, we can extend Theorem 1.2 as

Theorem 5.2.

$$(5.10) \quad s(9n) - 3s(n) = 2(1, 1, 3, 0, 0, 1)(n) - 4(4, 3, 4, 0, 4, 0)(n).$$

It is worthwhile to point out that Theorem 1.1 can be extended in a similar manner as

Theorem 5.3.

$$(5.11) \quad s(25n) - 5s(n) = 4(2, 2, 2, -1, 1, 1)(n) - 8(8, 3, 7, 2, 8, 4)(n).$$

Recalling (2.12), we see that all that is required to prove is

$$(5.12) \quad \begin{aligned} \phi(q)^3 - 3\phi(q)\phi(q^5)^2 &= 2 \sum_{x,y,z} q^{2x^2+2y^2+2z^2-yz+xz+xy} \\ &\quad - 4 \sum_{x,y,z} q^{8x^2+3y^2+7z^2+2yz+8xz+4xy}. \end{aligned}$$

It is easy to check that $8x^2 + 3y^2 + 7z^2 + 2yz + 8xz + 4xy$ can not represent any integer $\equiv 1, 2 \pmod{4}$. And so (5.11) reduces to (1.3) when $n \equiv 1, 2 \pmod{4}$. I leave the proof of (5.11) with $n \equiv 0, 3 \pmod{4}$ as an exercise for a motivated reader.

I now proceed to describe the generalization of the Theorem 1.2 for any odd prime p . Observe that the ternary quadratic form $x^2 + y^2 + 3z^2 + xy$ in this theorem has the discriminants 3^2 . We remind the reader that a discriminant of a ternary form $ax^2 + by^2 + cz^2 + dxy + ezx + fxy$ is defined as

$$\frac{1}{2} \det \begin{bmatrix} 2a & f & e \\ f & 2b & d \\ e & d & 2c \end{bmatrix}.$$

It is a well known fact (to those who know it well) that all ternary forms with the discriminant p^2 belong to the same genus, say $TG_{1,p}$. Let $|\text{Aut}(f)|$ denote the number of integral automorphs of a ternary quadratic form f , and let $R_f(n)$ denote the number of representations of n by f . Let p be an odd prime and $n \not\equiv 3 \pmod{4}$. I propose that

$$(5.13) \quad s(p^2n) - ps(n) = 48 \sum_{f \in TG_{1,p}} \frac{R_f(n)}{|\text{Aut}(f)|} - 96 \sum_{f \in TG_{1,p}} \frac{R_f\left(\frac{n}{4}\right)}{|\text{Aut}(f)|}.$$

Clearly, an inquiring mind wants to know if the parity restriction on n in (5.13) can be removed. In other words, the question is whether a straightforward generalization of Theorem 5.2 exists. Fortunately, the answer is a profound yes. However, the answer involves the second genus of ternary forms $TG_{2,p}$ with discriminant $16p^2$. Unfortunately, $TG_{2,p}$ is a bit hard to describe explicitly. Note that, in general, there are many genera of the ternary forms with the discriminant $16p^2$. However, when $p \equiv 3 \pmod{4}$ one can create $TG_{2,p}$ from some binary quadratic form of discriminant $-p$. Again, it is a well known fact that all binary forms with the discriminant $-p$ belong to the same genus, say BG_p . Let $ax^2 + bxz + cz^2$ be some binary form $\in BG_p$. We can convert it into ternary form

$$f(x, y, z) := 4ax^2 + py^2 + 4cz^2 + 4|b|xz.$$

Next, we extend f to a genus that contains f . This genus is, in fact, $TG_{2,p}$ when $p \equiv 3 \pmod{4}$. It can be shown that the map

$$BG_p \rightarrow TG_{2,p}$$

does not depend on which specific binary form from BG_p we have chosen as our starting point. I would like to comment that somewhat similar construction was employed in [2] to define the so-called S -genus. Let me illustrate this map for $p = 23$. In this case,

$$BG_{23} = \{x^2 + xz + 6z^2, 2x^2 + xz + 3z^2, 2x^2 - xz + 3z^2\}.$$

Choosing a binary form $x^2 + xz + 6z^2$ as a starting point one gets

$$\{x^2 + xz + 6z^2\} \rightarrow \{4x^2 + 23y^2 + 24z^2 + 4xz\} \rightarrow \\ \{4x^2 + 23y^2 + 24z^2 + 4xz, 8x^2 + 23y^2 + 12z^2 + 4xz, 3x^2 + 31y^2 + 31z^2 - 30yz + 2zx + 2xy\}.$$

We note that

$$TG_{2,23} := \{4x^2 + 23y^2 + 24z^2 + 4xz, 8x^2 + 23y^2 + 12z^2 + 4xz, 3x^2 + 31y^2 + 31z^2 - 30yz + 2zx + 2xy\}$$

is just one out of fifteen possible genera of the ternary form with the discriminant 8464. It is instructive to compare $TG_{2,23}$ and

$$TG_{1,23} := \{x^2 + 6y^2 + 23z^2 + xy, 2x^2 + 3y^2 + 23z^2 + xy, 3x^2 + 8y^2 + 8z^2 - 7yz + 2zx + 2xy\}.$$

Clearly,

$$|TG_{1,23}| = |TG_{2,23}|.$$

Moreover,

$$|\text{Aut}(3x^2 + 8y^2 + 8z^2 - 7yz + 2zx + 2xy)| = |\text{Aut}(3x^2 + 31y^2 + 31z^2 - 30yz + 2zx + 2xy)| = 12,$$

$$|\text{Aut}(x^2 + 6y^2 + 23z^2 + xy)| = |\text{Aut}(4x^2 + 23y^2 + 24z^2 + 4xz)| = 8,$$

$$|\text{Aut}(2x^2 + 3y^2 + 23z^2 + xy)| = |\text{Aut}(8x^2 + 23y^2 + 12z^2 + 4xz)| = 4.$$

It is a bit less obvious that

$$(3, 31, 31, -30, 2, 2)(4n) = (3, 8, 8, -7, 2, 2)(n),$$

$$(4, 23, 24, 0, 4, 0)(4n) = (1, 6, 23, 0, 0, 1)(n),$$

$$(8, 23, 12, 0, 4, 0)(4n) = (2, 3, 23, 0, 0, 1)(n),$$

and that

$$(3, 31, 31, -30, 2, 2)(m) = (4, 23, 24, 0, 4, 0)(m) = (8, 12, 23, 0, 0, 4)(m) = 0,$$

whenever $m \equiv 1, 2 \pmod{4}$. I propose that the above properties are, in fact, the signature properties of $TG_{2,p}$. In other words, for any odd prime p there exists an automorphism preserving bijection

$$H : TG_{2,p} \rightarrow TG_{1,p},$$

such that , for any $f \in TG_{2,p}$,

$$|\text{Aut}(f)| = |\text{Aut}H(f)|,$$

$$(5.14) \quad R_f(4n) = R_{H(f)}(n),$$

and

$$(5.15) \quad R_f(m) = 0, \quad \text{when } m \equiv 1, 2 \pmod{4}.$$

W.Jagy [13] suggested that $TG_{1,p} \cup TG_{2,p}$ does not represent any integer that is quadratic residue mod p when $p \equiv 1 \pmod{4}$, and when $p \equiv 3 \pmod{4}$ this union does not represent any integer that is a quadratic nonresidue mod p . That is for any $f \in TG_{1,p} \cup TG_{2,p}$

$$R_f(n) = 0,$$

when $(-n|p) = 1$. In addition, he pointed out that $TG_{2,p}$ represents a proper subset of those numbers represented by $TG_{1,p}$. Lastly, he observed that both $TG_{1,p}$ and $TG_{2,p}$ are anisotropic at p . I discuss one more example. This time I choose $p = 17$. Here one has

$$TG_{1,17} := \{3x^2 + 5y^2 + 6z^2 + yz + 2zx + 3xy, 3x^2 + 6y^2 + 6z^2 - 5yz + 2zx + 2xy\},$$

and

$$TG_{2,17} := \{7x^2 + 11y^2 + 20z^2 - 8yz + 4zx + 6xy, 3x^2 + 23y^2 + 23z^2 - 22yz + 2zx + 2xy\}.$$

Note that

$$\begin{aligned} |\text{Aut}(3x^2 + 5y^2 + 6z^2 + yz + 2zx + 3xy)| &= |\text{Aut}(7x^2 + 11y^2 + 20z^2 - 8yz + 4zx + 6xy)| = 4, \\ |\text{Aut}(3x^2 + 6y^2 + 6z^2 - 5yz + 2zx + 2xy)| &= |\text{Aut}(3x^2 + 23y^2 + 23z^2 - 22yz + 2zx + 2xy)| = 12, \\ (3, 23, 23, -22, 2, 2)(4n) &= (3, 6, 6, -5, 2, 2)(n), \\ (7, 11, 20, -8, 4, 6)(4n) &= (3, 5, 6, 1, 2, 3)(n), \\ (7, 11, 20, -8, 4, 6)(m) &= (3, 23, 23, -22, 2, 2)(m) = 0, \end{aligned}$$

whenever $m \equiv 1, 2 \pmod{4}$. It is worthwhile to point out that there are exactly fourteen genera with the discriminant 4624. Only three of those have the correct cardinality

$$\begin{aligned} |TG_{2,17}| &= 2, \\ |\{3x^2 + 6y^2 + 68z^2 + 2xy, \quad 10x^2 + 11y^2 + 14z^2 + 2yz + 4zx + 10xy\}| &= 2, \\ |\{5x^2 + 7y^2 + 34z^2 + 2xy, \quad 6x^2 + 12y^2 + 17z^2 + 4xy\}| &= 2. \end{aligned}$$

Note, however, that

$$|\text{Aut}(3x^2 + 6y^2 + 68z^2 + 2xy)| = |\text{Aut}(10x^2 + 11y^2 + 14z^2 + 2yz + 4zx + 10xy)| = 4,$$

and

$$|\text{Aut}(5x^2 + 7y^2 + 34z^2 + 2xy)| = |\text{Aut}(6x^2 + 12y^2 + 17z^2 + 4xy)| = 4.$$

And so, $TG_{2,17}$ is a unique genus with the desired properties.

I would like to conclude this discussion of $TG_{2,p}$ by providing a more explicit description valid in three special cases. If $p \equiv 3 \pmod{4}$, then $TG_{2,p}$ is the genus that contains

$$4x^2 + py^2 + (p+1)z^2 + 4zx.$$

I remark that the above form was obtained from the principal binary form $x^2 + xz + \frac{p+1}{4}z^2$. If $p \equiv 2 \pmod{3}$, then $TG_{2,p}$ is the genus that contains

$$x^2 + \frac{4p+1}{3}y^2 + \frac{4p+1}{3}z^2 + \frac{2-4p}{3}yz + 2zx + 2xy.$$

If $p \equiv 5 \pmod{8}$, then $TG_{2,p}$ is the genus that contains

$$8x^2 + \frac{p+1}{2}y^2 + (p+2)z^2 + 2yz + 8zx + 4xy.$$

Observe that the smallest prime to escape the above net of three special cases is $p = 73$. I am now ready to unveil the promised extension of (5.13). Let p be an odd prime, then

$$(5.16) \quad s(p^2n) - ps(n) = 48 \sum_{f \in TG_{1,p}} \frac{R_f(n)}{|\text{Aut}(f)|} - 96 \sum_{f \in TG_{2,p}} \frac{R_f(n)}{|\text{Aut}(f)|}.$$

The proof of this neat formula with $p \geq 7$ is beyond the scope of this paper and will be given elsewhere. Note, that (5.13) follows easily from (5.14), (5.15) and (5.16).

I would like to conclude with the following soothing examples

$$(5.17) \quad s(7^2n) - 7s(n) = 6(1, 2, 7, 0, 0, 1)(n) - 12(4, 7, 8, 0, 4, 0)(n),$$

$$(5.18) \quad \begin{aligned} s(11^2n) - 11s(n) &= 4(3, 4, 4, -3, 2, 2)(n) + 6(1, 3, 11, 0, 0, 1)(n) \\ &\quad - 8(3, 15, 15 - 14, 2, 2)(n) - 12(4, 11, 12, 0, 4, 0)(n), \end{aligned}$$

$$(5.19) \quad s(13^2n) - 13s(n) = 12(2, 5, 5, -3, 1, 1)(n) - 24(8, 7, 15, 2, 8, 4)(n),$$

$$(5.20) \quad \begin{aligned} s(17^2n) - 17s(n) &= 12(3, 5, 6, 1, 2, 3)(n) + 4(3, 6, 6, -5, 2, 2)(n) \\ &\quad - 24(7, 11, 20, -8, 4, 6)(n) - 8(3, 23, 23, -22, 2, 2)(n), \end{aligned}$$

$$(5.21) \quad \begin{aligned} s(19^2n) - 19s(n) &= 6(1, 5, 19, 0, 0, 1)(n) + 12(4, 5, 6, 5, 1, 2)(n) \\ &\quad - 12(4, 19, 20, 0, 4, 0)(n) - 24(7, 11, 23, -10, 6, 2)(n), \end{aligned}$$

$$(5.22) \quad \begin{aligned} s(23^2n) - 23s(n) &= 4(3, 8, 8, -7, 2, 2)(n) + 6(1, 6, 23, 0, 0, 1)(n) \\ &\quad + 12(2, 3, 23, 0, 0, 1)(n) - 8(3, 31, 31, -30, 2, 2)(n) \\ &\quad - 12(4, 23, 24, 0, 4, 0)(n) - 24(8, 23, 12, 0, 4, 0)(n), \end{aligned}$$

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